

## Midterm

The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could lose coherent narrative through line. If he can't make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics.

### Problem 1 :

By using the multiplication table, show that there is only one (up to isomorphism) group of order 3.

*	$e$	$a$	$b$
$e$			
$a$			
$b$			

### Problem 2 :

- 1.(a) Give the definition of a cyclic group.
- (b) Give the definition of an commutative group.
- (c) Show that every cyclic group is commutative.
2. Consider the group  $\mathbb{Z}/16\mathbb{Z}$  of residues modulo 16 (under addition modulo 16). How many subgroups does this group have? Explain your answer.

### Problem 3 :

Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z} \text{ with } a = \pm 1 \right\}$$

1. Show that  $G$  is a group under matrix multiplication.
2. Is  $G$  abelian? Explain your answer.
3. Describe all elements of order two in  $G$ .

**Problem 4 :**

1. State Lagrange's theorem.
2. Show that every group of prime order is cyclic.
3. The set of ordinary integers  $\mathbb{Z}$  is a subgroup of the additive group of rational numbers  $\mathbb{Q}$ . Show that  $\mathbb{Z}$  has infinite index in  $\mathbb{Q}$  (that is, there are infinitely many (left or right) cosets of  $\mathbb{Z}$  in  $\mathbb{Q}$ ).

**Problem 5 :**

Let  $G$  be a finite group,  $X$  be a set and  $G \times X \rightarrow X$  be a group action. Let  $x_0 \in X$ .

1. Give the definition of the stabilizer  $Stab(x_0)$  of  $x_0$ .
2. Give the definition of the orbit  $O(x_0)$  of  $x_0$ .
3. Prove that the map

$$(1) \quad \psi : G/Stab(x_0) \rightarrow O(x_0) \quad \text{defined by} \quad \psi(g \cdot Stab(x_0)) = g \cdot x_0$$

is a well define map.

4. Prove that  $\psi$  is a bijection.
5. Deduce that the size  $|O_{x_0}|$  of any individual orbit must divide  $|G|$ .